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Control of motion of an inhomogeneous cylinder with internal movable masses along a horizontal plane☆

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Abstract

The controlled motion of a rigid inhomogeneous cylinder over a rough horizontal plane is considered. The control is provided by controlled motion of internal masses. Mathematical models are constructed that correspond to rolling without loss of contact or slippage. The conditions for the physical implementability of such a motion are derived. The case where the internal moving masses from a rigid flywheel the centre of inertia of which lies on the axis of the cylinder is investigated in detail. A near-time-optimal feedback control that enables the total energy to be changed in a required way is constructed on the basis of an asymptotic approach. The main operating modes are simulated, namely, swinging up of the cylinder to a large angular amplitude, rotation with a prescribed energy, deceleration of rolling to a complete stop, and oscillations and rotations in the neighbourhood of the separatrix.

Among the classical models of systems considered in mechanics and physics, oscillating and rotating systems, the simplest of which is a simple, or physical pendulum, play an important role. These models enable one to investigate important properties and features of physical and engineering systems of different kinds, such as the presence and stability of steady states, oscillating and rotating modes of motion, and the features of the transition from one to the other, and methods for exciting or damping oscillations and rotations. An important aspect of such investigation is the study of the possibility of controlling oscillations and rotations and of constructing specific control laws which enable the system to be brought to the required state, subject to constraints accounted the physical features of the system (for example, the nature of the constraints – bilateral or unilateral) or by the operational life of the control system and drives.

A large number of publications have been devoted to the control of oscillating and rotating systems. Asymptotic methods have been proposed and developed for solving optimal control problems for oscillating the systems equations of motion of which involve a small parameter.[1,2] These methods are based on Pontryagin's maximum principle[3] combined with the Krylov–Bogolyubov averaging method.[4,5] Using the averaging method, a large number of specific control problems have been solved for non-linear pendulum-type systems. The moment of forces applied to the fixed axis of rotation of the pendulum, the force applied to the moving massive suspension point or the velocity of the suspension point are used as the control variable. A wide class of problems of controlling the displacement of linear pendulum-type systems, having important engineering applications, for example, in the design of automatic control systems for cranes and container loaders, have been considered.[1]

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Problems of the optimal parametric control of a pendulum with a moving point body inside it have been solved.[6,7] The point body can move along a straight line, passing through the suspension point and the centre of mass of the pendulum; the control variable is the displacement[6] or velocity[7] of the internal mass with respect to the pendulum, the range of variation of which is bounded. The optimal control of the motion of a plane-parallel bifilar pendulum due to displacement of an internal point body has been considered.[8] The components of the velocity or acceleration of motion of an internal point mass relative to the body of the pendulum are the controls. The results obtained enable the potential for exciting or damping the vibrations in swing-type mechanisms to be clarified.

The control and stabilization of a physical pendulum suspended on a fixed horizontal axis, with a massive flywheel inside it was investigated in Ref. 9. The flywheel can rotate about a horizontal axis, fixed with respect to the pendulum, and is set in motion by an electric drive. A control law was constructed for the electric motor of the drive, according to which the pendulum is brought from any initial state to the upper unstable equilibrium position and is stabilized in this position. Modes for controlling the rotations of the pendulum using the flywheel were also examined. The theoretical results were confirmed by experiments. The control method is in fact parametric, since, like the systems mentioned above,[6–8] the pendulum is controlled by moving a body inside it.

Below we consider the controlled rolling over a horizontal plane of a rigid cylinder with a flywheel inside it. The centre of mass of the cylinder in the general case does not lie on its geometric axis (the cylinder is out-of-balance), but the centre of mass of the flywheel lies on this axis. Control is provided by changing the angular acceleration of the flywheel with respect to the cylinder. The system can be classified as an oscillational-rotational one, extends the well-known model of controlled motion of a pendulum, and by its nature is similar to that examined earlier.[9] In both cases the carrying body (the pendulum) is controlled by controlling the rotation of the flywheel inside it. Unlike the case of a fixed axis of the pendulum suspension,[9] here the line of contact of the wheel with the supporting plane is moving, as the wheel is rolling over a plane. Furthermore, the rolling conditions of the cylinder over the plane represent a unilateral constraint, on account of which it is necessary to impose additional conditions, which enable the non-violation of this constraint to be monitored.

The results obtained are of interest from the viewpoint of the theory of the control of oscillational-rotational systems, and also in relation to possible applications in robotics and the mechanics of transport mechanisms, in particular, monocycles.[10,11]

1. Initial mechanical model

We will consider the plane-parallel motion (rolling) of an inhomogeneous perfectly rigid circular cylinder over a rough horizontal plane. Inside the cylinder there is a set of point masses that can move with respect to it in a controlled way. In the plane perpendicular to the axis of the cylinder and passing through its centre of mass, we will introduce a fixed (inertial) system of coordinates XY with its centre at the point O and a mobile system of coordinates $\xi\eta$, rigidly connected to the cylinder, with its centre at the point o on the axis of the cylinder (Fig. 1). The X axis is selected as the line of intersection of the coordinate plane XY with the horizontal plane along which the cylinder is rolling, and the Y axis is directed vertically upwards. The axis η passes through the centre of mass C of the cylinder and is directed from point C to point o (Fig. 1).

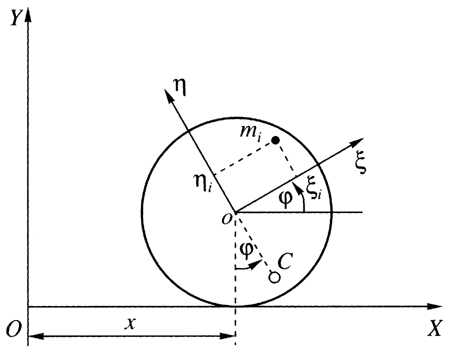

Fig. 1.

We will assume that, during the motion, no slippage of the cylinder and/or loss of contact with the plane occurs. In this case the position of the cylinder is characterized by a single variable (generalized coordinate) x – the coordinate of the projection of the point o onto the X axis, or the angle φ of rotation of the cylinder about its axis. We will define φ as the angle between the ξ and X axes, considering the counter clockwise rotation of the ξ axis to be positive. The variables x and ξ are related by the equation

$$x - x_0 = -R(\varphi - \varphi_0) \tag{1.1}$$

where R is the radius of the cylinder, and x_0 and φ_0 are constants determined by the choice of the reference points for the corresponding variables. Without loss of generality, it is possible to assume that $x_0 = 0$ and $\varphi_0 = 0$. The position of a point mass m_i ($i = 1, \ldots, n$) with respect to the cylinder is described by the Cartesian coordinates ξ_i and η_i in the $\xi\eta$ system. Below, the functions $\xi_i(t)$ and $\eta_i(t)$ are considered to be specified and sufficiently smooth.

If the angle φ is selected as the generalized coordinate, the expressions for the kinetic energy T_M and potential energy U_M of the cylinder have the form

$$T_M = \frac{1}{2}[J_C + M(R^2 + l^2 - 2Rl\cos\varphi)]\dot{\varphi}^2, \quad U_M = Mgl(1 - \cos\varphi) \tag{1.2}$$

where J_C is the moment of inertia of the cylinder about the axis passing through the centre of mass C parallel to the generatrix, M is the mass of the cylinder, l is the distance from the axis of the cylinder to its centre of mass and g is the acceleration due to gravity.

The kinetic energy T_m and the potential energy U_m of the set of point masses m_i are equal to the sums of the corresponding energies T_{mi} and U_{mi} of the individual points

$$\begin{gathered}
T_m = \sum_{i=1}^{n} T_{mi}, \quad U_m = \sum_{i=1}^{n} U_{mi} \\
T_{mi} = \frac{m_i^* R^2 \dot{\varphi}^2}{2} + (m_{i\xi}\dot{\xi}_i + m_{i\eta}\dot{\eta}_i)R\dot{\varphi} + \frac{m_i(\dot{\xi}_i^2 + \dot{\eta}_i^2)}{2} \\
m_i^* = m_i\left[1 + \frac{(\xi_i^2 + \eta_i^2)}{R^2} + 2\left(\frac{\xi_i}{R}\right)\sin\varphi + 2\left(\frac{\eta_i}{R}\right)\cos\varphi\right] \\
m_{i\xi} = -m_i\left(\frac{\eta_i}{R} + \cos\varphi\right), \quad m_{i\eta} = m_i\left(\frac{\xi_i}{R} + \sin\varphi\right) \\
U_{mi} = m_i g(\xi_i \sin\varphi + \eta_i \cos\varphi) + U_{i0}, \quad U_{i0} = \text{const}
\end{gathered} \tag{1.3}$$

If the motion of the points m_i relative to the cylinder is specified by sufficiently smooth functions $\xi_i(t)$ and $\eta_i(t)$, the motion of the cylinder is described by Lagrange's equation

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{\varphi}} - \frac{\partial T}{\partial \varphi} = -\frac{\partial U}{\partial \varphi}, \quad T = T_M + T_m, \quad U = U_M + U_m \tag{1.4}$$

In deriving the equations of motion, relation (1.1), implying that rolling of the cylinder over the plane occurs without loss of contact or slippage, was postulated. In fact satisfaction of this condition is ensured by the physical forces applied on the contact line – the normal reaction of the supporting surface N, directed along the Y axis, and the friction force F directed along the X axis of the inertial reference system XY. The normal reaction should be non-negative, since the supporting surface can act on the cylinder only towards the latter. We will assume that dry friction, which obeys Coulomb's law, acts between the cylinder and the supporting surface. In this case, the magnitude of the friction force cannot exceed the magnitude of the normal reaction multiplied by the coefficient of friction. Thus, the complete set of conditions of motion of the cylinder over the plane without loss of contact or slippage, taking the physical features of the active forces into account, has the form

$$\dot{x} + R\dot{\varphi} \equiv 0, \quad y \equiv R, \quad N \geq 0, \quad |F| \leq kN \tag{1.5}$$

where x and y are the X and Y coordinates of the point o of intersection of the cylinder with the XY coordinate plane, and k is the coefficient of friction. Note that the inequality $N \geq 0$ stems from the inequality $|F| \leq kN$, and hence the first inequality in system (1.5) can be omitted.

In the case of motion without loss of contact or slippage, the forces N and F are defined by the expressions

$$\begin{aligned} N &= (M+m)g + Ml(\ddot{\varphi}\sin\varphi + \dot{\varphi}^2\cos\varphi) + \\ &+ \sum_{i=1}^{n} m_i[\ddot{\xi}_i\sin\varphi + \ddot{\eta}_i\cos\varphi + \ddot{\varphi}(\xi_i\cos\varphi - \eta_i\sin\varphi) - \\ &- \dot{\varphi}^2(\xi_i\sin\varphi + \eta_i\cos\varphi) + 2\dot{\varphi}(\dot{\xi}_i\cos\varphi - \dot{\eta}_i\sin\varphi)] \end{aligned} \tag{1.6}$$

$$\begin{aligned} F &= (M+m)\ddot{x} + Ml(\ddot{\varphi}\cos\varphi - \dot{\varphi}^2\sin\varphi) + \\ &+ \sum_{i=1}^{n} m_i[\ddot{\xi}_i\cos\varphi - \ddot{\eta}_i\sin\varphi - \ddot{\varphi}(\xi_i\sin\varphi + \eta_i\cos\varphi) - \\ &- \dot{\varphi}^2(\xi_i\cos\varphi - \eta_i\sin\varphi) - 2\dot{\varphi}(\dot{\xi}_i\sin\varphi + \dot{\eta}_i\cos\varphi)] \end{aligned} \tag{1.7}$$

where m is the sum of the masses of the point masses m_i.

Expressions (1.6) and (1.7) are derived on the basis of the centre of mass principle for a mechanical system. The current coordinates X_C and Y_C of the centre of mass of the system under consideration in the fixed reference system XY are defined by the expressions

$$\begin{aligned} (M+m)X_C &= M(x + l\sin\varphi) + \sum_{i=1}^{n} m_i(x + \xi_i\cos\varphi - \eta_i\sin\varphi) \\ (M+m)Y_C &= M(R - l\cos\varphi) + \sum_{i=1}^{n} m_i(R + \xi_i\sin\varphi + \eta_i\cos\varphi) \end{aligned} \tag{1.8}$$

Of the external forces, the force of gravity, equal to $(M+m)g$ and directed vertically downwards, and the normal reaction N act on the system along the Y axis, and only the friction force F acts on the system along the X axis. Accordingly, the equations of motion of the centre of mass have the form

$$(M+m)\ddot{X}_C = F, \quad (M+m)\ddot{Y}_C = -(M+m)g + N \tag{1.9}$$

Eqs. (1.6) and (1.7) follow from relations (1.8) and (1.9).

From the equation of motion (1.4) we can express the generalized acceleration $\ddot{\varphi}$ in terms of the phase variables (the generalized coordinate φ and the generalized velocity $\dot{\varphi}$) and substitute the expressions obtained into relations (1.6) and (1.7). As a result, we obtain the normal reaction and the friction force as functions of the phase variables of the system and time.

Motion of the cylinder over the plane occurs without loss of contact or slippage if, and only if, at any instant of time t, conditions (1.5) satisfied.

From relations (1.2)–(1.4) it can be seen that the functions $\xi_i(t)$, $\dot{\xi}_i(t)$, $\ddot{\xi}_i(t)$, $\eta_i(t)$, $\dot{\eta}_i(t)$ and $\ddot{\eta}_i(t)$, which characterize the motion of points m_i relative to the cylinder, occur in the equation of motion (1.4). By controlling the motion of these points, it is possible to control the motion of the cylinder. In the formulation of control problems, it is reasonable to take $\ddot{\xi}_i(t)$ and $\ddot{\eta}_i(t)$ as the control variables.

In the special case when the coordinates ξ_i and η_i are constant, the system under consideration comprises a rigid cylinder with a corresponding mass distribution. Such a system, depending on the initial conditions, performs vibrational or rotational motions and can be treated as a generalized physical pendulum whose motion is described by an equation of the form

$$J_*(\varphi)\ddot{\varphi} + \frac{J'_*(\varphi)\dot{\varphi}^2}{2} + U'_*(\varphi) = 0 \tag{1.10}$$

where $J_*(\varphi)$ is the moment of inertia of the system about the instantaneous contact line (the generatrix of the cylinder), $U_*(\varphi)$ is the potential energy of the system in the gravitational field, and the prime denotes differentiation with respect to the variable φ. System (1.10) has the energy integral

$$\frac{J_*(\varphi)\dot{\varphi}^2}{2} + U_*(\varphi) = \text{const}$$

and allows of complete integration in terms of elliptic functions.

The equation of motion (1.4), with an arbitrary number of point masses moving inside the cylinder, and with arbitrary functions $\xi(t)$ and $\eta(t)$, is rather cumbersome and complex for solving control problems for such systems. It is therefore reasonable to impose constraints on the motion of these points in order to simplify the equation of motion.

2. Simplified versions of the model

2.1. Specified trajectories of motion of the point masses

In many applied problems it is reasonable to assume that the point masses m_i move relative to the cylinder along prescribed trajectories. Mechanically, such motion can be ensured, for example, by guides of appropriate shape. The trajectories of motion in the $\xi\eta$ system of coordinates can be represented in the form of curves specified parametrically as follows:

$$\xi_i = \xi_i(s_i), \quad \eta_i = \eta_i(s_i), \quad i = 1, 2, \ldots, n \tag{2.1}$$

where $\xi_i(s_i)$ and $\eta_i(s_i)$ are specified functions defining the shape of the curve, and s_i is a parameter of the curve, for example, its length, that is being measured from some fixed point. Characterizing the position of a point m_i on the corresponding trajectory by the parameter s_i, it is possible to represent the motion of this point relative to the cylinder by the functions $\xi_i(s_i(t))$ and $\eta_i(s_i(t))$, where $s_i(t)$ represents the variation of the parameter of the curve with time during the motion of the point. The derivatives of the functions $s_i(t)$ can be regarded as control variables.

Some of the simplest examples of curves (2.1), which at the same time are of importance in research and applied respects, are circles with a common centre at the point o on the axis of the cylinder. In this case, Eq. (2.1) are defined specifically as follows:

$$\xi_i = \rho_i\cos\psi_i, \quad \eta_i = \rho_i\sin\psi_i, \quad \psi_i = \psi_i(t); \quad 0 \le \rho_i \le R, \quad \rho_i = \text{const} \tag{2.2}$$

where ρ_i represents the radii of the circles, and ψ_i is the angle between the radius vector of the point on the corresponding circle and the ξ axis. The angular variables ψ_i in expressions (2.2) play the role of the parameters s_i in relations (2.1).

Substituting expressions (2.2) into relations (1.3), we obtain comparatively simple expressions for the kinetic and potential energies of the point masses

$$\begin{aligned} T_{mi} &= \frac{1}{2}m_i[R^2\dot{\varphi}^2 + \rho_i^2(\dot{\varphi} + \dot{\psi}_i)^2 + 2R\rho_i\dot{\varphi}(\dot{\varphi} + \dot{\psi}_i)\sin(\varphi + \psi_i)] \\ U_{mi} &= m_i g\rho_i\sin(\varphi + \psi_i) + U_{i0}; \quad i = 1, 2, \ldots, n \end{aligned} \tag{2.3}$$

Lagrange's equation (1.4) with this simplification still remains cumbersome and complex for a qualitative analysis and the solution of control problems. In solving a number of control problems (including the optimal one) for systems of this kind, it is reasonable to take the functions $\ddot{\psi}_i$ as control variables, assuming them to be piecewise continuous. Then the variables $\dot{\psi}_i$ and ψ_i are found by integrating the control function, and the function $\psi_i(t)$ will be continuously differentiable, and $\dot{\psi}_i(t)$ will be piecewise differentiable, which corresponds to physical concepts concerning the nature of the variation of the coordinate and velocity of a point body. At the same time, the variables ψ_i and $\dot{\psi}_i$ for a controlled system treated in this way are phase variables, and, if there are many point masses, it is necessary to solve a control problem with a large number of phase variables, which is rather difficult.

2.2. The dynamic balance of internal moving masses

Let the number of internal moving masses m_i be even ($n=2j^*$) and let the point masses with numbers j and $j+j^*$ ($j=1,\ldots,j^*$) have identical masses and, at each instant of time, be symmetrical about the centre o. In this case we have

$$\begin{aligned}
&m_j = m_{j+j^*}, \quad \xi_{j+j^*} = -\xi_j, \quad \eta_{j+j^*} = -\eta_j \\
&T_{mj} + T_{m,j+j^*} = m_j(R^2+\xi_j^2+\eta_j^2)\dot{\varphi}^2 + \\
&+2m_j(\xi_j\dot{\eta}_j - \eta_j\dot{\xi}_j)\dot{\varphi} + m_j(\dot{\xi}_j^2+\dot{\eta}_j^2) \\
&U_{mj} + U_{m,j+j^*} = 2U_{j0} = \text{const}; \quad j = 1, \ldots, j^*
\end{aligned} \tag{2.4}$$

If the conditions imposed are satisfied, the centre of mass of the system of point masses m_i lies on the axis of the cylinder (at point o). Therefore, in this case it is appropriate to call the system of internal moving masses dynamically balanced.

Expressions (2.4) do not contain linear terms in ξ_j, $\dot{\xi}_j$ and η_j, $\dot{\eta}_j$. Furthermore, there is no dependence on φ, which considerably simplifies the equations of controlled motion. If, furthermore, the points with numbers j and $j+j^*$ ($j=1,\ldots,j^*$) move synchronously in circles, expression (2.4) reduces to the form

$$T_{mj} + T_{m,j+j^*} = m_jR^2\dot{\varphi}^2 + m_j\rho_j^2(\dot{\varphi}+\dot{\psi}_j)^2 \tag{2.5}$$

where ρ_j and ψ_j are the parameters of the circle, defined by relations (2.2).

When the system of internal masses is dynamically balanced and the corresponding point masses move in circles, the equation of motion (1.4) acquires the comparatively simple obvious form

$$\begin{aligned}
&J_*(\varphi)\ddot{\varphi} + Ml(g+R\dot{\varphi}^2)\sin\varphi = -2\sum_{j=1}^{j^*} m_j\rho_j^2\ddot{\psi}_j \\
&J_*(\varphi) = J_C + J_m + (M+m)R^2 + Ml^2 - 2MlR\cos\varphi \\
&m = \sum_{i=1}^{n} m_i, \quad J_m = 2\sum_{j=1}^{j^*} m_j\rho_j^2
\end{aligned} \tag{2.6}$$

where J_m is the total moment of inertia of the point masses m_j about the axis of the cylinder, and $J_*(\varphi)$ is the overall moment of inertia of the cylinder with moving masses about the instantaneous axis of rotation of the cylinder (the generatrix along which the cylinder touches the plane on which rolling occurs.

Let all the point masses m_i move in their circles with the same angular velocity, i.e. $\dot{\psi}_j \equiv \dot{\psi}$ for all $j=1,\ldots,j^*$. In such a case, the moving masses form a rigid body (flywheel) rotating relative to the cylinder about the axis of the latter. Here, the equation of motion (2.6) is simplified even further and takes the form

$$J_*(\varphi)\ddot{\varphi} + Ml(g+R\dot{\varphi}^2)\sin\varphi = -J_m\ddot{\psi} \tag{2.7}$$

Expressions (1.6) and (1.7) for the normal component N and the tangential component F of the reaction of the supporting surface in the case under consideration have the form

$$\begin{aligned}
&N = (M+m)g + Ml(\ddot{\varphi}\sin\varphi + \dot{\varphi}^2\cos\varphi) = (M+m)g + \\
&+ Ml\dot{\varphi}^2\cos\varphi - MlJ_*^{-1}\sin\varphi[J_m\ddot{\psi} + Ml(g+R\dot{\varphi}^2)\sin\varphi]
\end{aligned} \tag{2.8}$$

$$\begin{aligned}
&F = Ml(\ddot{\varphi}\cos\varphi - \dot{\varphi}^2\sin\varphi) - (M+m)R\ddot{\varphi} = -Ml\dot{\varphi}^2\sin\varphi + \\
&+ J_*^{-1}[(M+m)R - Ml\cos\varphi][J_m\ddot{\psi} + Ml(g+R\dot{\varphi}^2)\sin\varphi]
\end{aligned} \tag{2.9}$$

In accordance with relations (1.5), the magnitudes of N and F must satisfy the conditions $N \geq 0$ and $|F| \leq kN$, where k is the coefficient of friction. If there is no imbalance of the cylinder ($l=0$), the first inequality is always satisfied, and the second inequality is satisfied for a sufficiently small magnitude of the relative angular acceleration of the pendulum or a sufficiently high coefficient of friction, namely, if $J_m J_*^{-1} R|\ddot{\psi}| \leq kg$. If there is an imbalance, both inequalities may be violated for sufficiently large $|\dot{\varphi}|(\lambda\dot{\varphi}^2\ g)$ or $|\ddot{\psi}|(J_m J_*^{-1} l|\ddot{\psi}|\ g)$. In the space of the variables l, $\dot{\varphi}$ and $\ddot{\psi}$ there is a region in which inequalities (1.5) are satisfied for any φ.

When there is no imbalance ($l = 0$), we have the equation

$$J_*\dot{\varphi} + J_m\dot{\psi} = \text{const}$$

which expresses the conservation of the angular momentum of the system about the line of contact of the cylinder with the plane. If the entire system was at rest ($\dot{\varphi} = 0$, $\dot{\psi} = 0$) at the initial instant of time, the constant on the right-hand side of this equation is equal to zero. In this case, for the cylinder to roll in a particular direction, the flywheel must the whole time rotate relative to the cylinder in the opposite direction to cylinder rotation during rolling, i.e. the magnitude of $\dot{\psi}$ must be a constant in sign and not vanish. Note that the acceleration of the cylinder to any velocity is possible for an arbitrarily small value of $|\ddot{\psi}|$, which makes it possible to avoid slippage during acceleration. If $l > 0$, acceleration of the cylinder from the stable equilibrium state (when the centre of mass of the cylinder lies on a vertical line passing through the axis of the cylinder, below the latter) is prevented by the moment of the forces of gravity $-Mgl\sin\varphi$ and the moment of the centrifugal forces of inertia $-MlR\dot{\varphi}^2\sin\varphi$. This can be seen from Eq. (2.7). Nonetheless, in this case, also, acceleration of the cylinder is possible for an arbitrarily small value of $|\ddot{\psi}|$ due to resonance swinging of the system. To avoid violations of the conditions for rolling without loss of contact and slippage (1.5), the magnitude of the imbalance l and the magnitude of the relative angular acceleration of the flywheel $\ddot{\psi}$ must be sufficiently small.

As can be seen from relation (2.7), when $l > 0$ and $\ddot{\psi} = 0$, the cylinder on the plane has a denumerable set of stable equilibrium positions $\varphi_e = 2\pi p$ ($x_e = -2R\pi p$) ($p = 0, \pm 1, \ldots$), which may be of interest for applications.

Below, optimal control problems are solved for system (2.7), which seems to be the simplest system of the class being considered. The quantity $\ddot{\psi}$ is used as the control variable. The performance index in such problems is normally the time needed to bring the system to the prescribed phase state or the total mechanical energy at the instant the control process is completed. By virtue of the above, to satisfy the conditions of motion of the cylinder without loss of contact with the plane or slippage, the value of $|\ddot{\psi}|$ must be sufficiently small. We will therefore consider $|\ddot{\psi}|$ to be an asymptotically small quantity, which will enable us to use effective asymptotic methods for the approximate construction of optimal motions.

3. The construction of a standard system of equations of the controlled rolling of a cylinder

Assuming $M \neq 0$ and $l \neq 0$, we will introduce the following dimensionless variables and parameters

$$\theta = \nu t, \quad \nu = \sqrt{\frac{Mlg}{J}}, \quad \lambda = \frac{MRl}{J}, \quad u = \frac{\ddot{\psi}}{\gamma}, \quad \varepsilon = \frac{J_m\gamma}{J\nu^2}$$
$$J = J_C + J_m + (M+m)R^2 + Ml^2 \tag{3.1}$$

and represent Eq. (2.7) in the form

$$(1 - 2\lambda\cos\varphi)\varphi'' + (1 + \lambda\varphi'^2)\sin\varphi = -\varepsilon u, \quad \varepsilon \ll 1, \quad |u| \leq 1 \tag{3.2}$$

and the conditions of rolling without loss of contact or slippage (1.5), taking expressions (2.8) and (2.9) for N and F into account, in the form

$$N^* \geq 0, \quad |F^*| \leq kN^*$$
$$N^* = \zeta + \lambda^2\left[\varphi'^2\cos\varphi - \frac{\varepsilon u + (1 + \lambda\varphi'^2)\sin\varphi}{1 - 2\lambda\cos\varphi}\sin\varphi\right]$$
$$F^* = -\lambda^2\varphi'^2\sin\varphi + \lambda(\zeta - \cos\varphi)\frac{\varepsilon u + (1 + \lambda\varphi'^2)\sin\varphi}{1 - 2\lambda\cos\varphi} \tag{3.3}$$
$$\zeta = \frac{M+m}{J}R^2$$

The prime denotes differentiation with respect to the dimensionless time θ, and γ is the maximum admissible value of the magnitude of the angular acceleration of the flywheel $\ddot{\psi}$ (a control system parameter). The parameter ν characterizes the frequency $\nu_e = \nu/(\sqrt{1-2\lambda})$ of small oscillations of the cylinder in the neighbourhood of the equilibrium positions $\varphi_e = 0, \pm 2\pi, \ldots$ when there is no control.

The dimensionless parameter λ lies in the range $0 < \lambda < ½$. To prove this inequality, we will represent the overall moment of inertia of the system about the instantaneous line of contact of the cylinder with the plane, defined by the second equation of system (2.6), in the form $J_*(\varphi) = J(1 - 2\lambda \cos\varphi)$. The value of $J_*(\varphi)$ is positive for all φ. The inequality $\lambda < ½$ follows from the relation $J_*(0) > 0$.

If $M = 0$ or $l = 0$, replacement of the variables (3.1) is inapplicable due to the fact that the parameter ν, which occurs in the formula relating the initial (dimensional) time t to the dimensionless time θ, vanishes. Therefore, the cases indicated require a separate examination. From relations (2.6) and (2.7) it follows that, when $M = 0$ or $l = 0$, Eq. (2.7) acquires the form

$$J^*\ddot{\varphi} = -J_m\ddot{\psi}, \quad J^* = \begin{cases} J_C + J_m + (M+m)R^2, & \text{если} \quad l = 0 \\ J_m + mR^2, & \text{если} \quad M = 0 \end{cases} \tag{3.4}$$

We have taken into account the fact that, when the mass of the cylinder M becomes zero, its moment of inertia J_C also becomes zero.

Formally, Eq. (3.4) is identical with the equation of motion of a point mass along a horizontal straight line under the action of the control force. The coordinate of the point mass is identified with the variable φ, the mass is identified with the constant coefficient J^* and the control force is identified with the quantity $-J_m\ddot{\psi}$. The control of this system has been investigated in great detail and is not considered in the present paper.

Note that, if in Eq. (3.2) it is assumed that $\lambda = 0$, an equation will be obtained that formally identical with the equation of motion of a pendulum (simple or physical) with a fixed horizontal axis of rotation under the action of a control moment of forces $-\varepsilon u$ applied to this axis. In terms of the initial dimensional variables, this case corresponds to $R = 0$ with $l \neq 0$. Such a situation is physically possible since, in deriving expressions (1.2) and (1.3) for the kinetic energy and potential energy of the system, it is important only that the surface of the body, with moving masses inside, rolling over the plane, cylindrical, while the shape of the body itself and the distribution of masses inside it are unimportant. Therefore, the radius of the rolling cylinder R and the distance from the axis of this cylinder to the centre of mass of the body carrying the moving masses are in no way related. Thus, the system considered extends the model of a pendulum controlled by the moment of forces applied to its axis of rotation.

When $\varepsilon = 0$ or $u = 0$, there is no control. In this case, system (3.2) has a first integral (energy the integral)

$$E = \frac{1}{2}(1 - 2\lambda\cos\varphi)\varphi'^2 + 1 - \cos\varphi \tag{3.5}$$

From the physical viewpoint, the quantity E is the total mechanical energy of the unbalanced cylinder with a flywheel rigidly fixed to it, i.e. the quantity E does not take into account the components of the kinetic energy related to rotation of the flywheel with respect to the cylinder. If $E < 2$, the system performs periodic oscillations about the position of stable equilibrium, with the amplitude $\varphi_* = \arccos(1 - E)$. If $E > 2$, the cylinder rolls (rotates) with a periodically varying velocity. The case when $E = 2$ corresponds to the separatrix.

The oscillation period Θ_υ and the rotation period Θ_r are determined in a standard way for a conservative system[1,2]

$$\begin{aligned} &\Theta_\upsilon(E, \lambda) = 2\sqrt{2}I_\upsilon(\varphi_*, 2\lambda), \quad \varphi_* = \arccos(1 - E), \quad 0 < E < 2 \\ &I_\upsilon(\varphi_*, \alpha) = \int_0^{\varphi_*}\left(\frac{1 - \alpha\cos\varphi}{\cos\varphi - \cos\varphi_*}\right)^{\frac{1}{2}} d\varphi, \quad 0 < \alpha < 1 \end{aligned} \tag{3.6}$$

$$\Theta_r(E,\lambda) = \sqrt{2\beta} I_r(\beta, 2\lambda), \quad \beta = \frac{1}{(E-1)}, \quad E > 2$$
$$I_r(\beta,\alpha) = \int_0^{\pi}\left(\frac{1-\alpha\cos\varphi}{1+\beta\cos\varphi}\right)^{\frac{1}{2}} d\varphi, \quad 0<\alpha<1, \quad 0<\beta<1 \tag{3.7}$$

Integrals of the form $I_\upsilon(\varphi_*, \alpha)$ and $I_r(\beta, \alpha)$ are not given in the most comprehensive reference books and require separate calculations. They can be expressed in terms of a complete elliptic integral of the third kind $\Pi((\pi/2), p, \kappa)$.[12]

After a non-trivial change of the argument φ and lengthy calculations, the function $I_\upsilon(\varphi_*, \alpha)$ (3.6) can be represented in the form

$$I_\upsilon(\varphi_*,\alpha) = \frac{\sqrt{2}(1-\alpha)}{\sqrt{1-\alpha\cos\varphi_*}}\Pi\left(\frac{\pi}{2}, p, \kappa\right)$$
$$p = \frac{2\alpha}{\alpha+1}\kappa^2, \quad \kappa = \sqrt{\frac{(1+\alpha)(1-\cos\varphi_*)}{2(1-\alpha\cos\varphi_*)}} < 1 \tag{3.8}$$

Note that the coefficient of the integral $\Pi((\pi/2), p, \kappa)$ in Eq. (3.8) approaches zero as $\alpha \to 1$. However, in this case the modulus κ and the parameter p of the integral approach unity from below, which leads to its divergence ($\Pi\ (1-\alpha)^{-1}$). At the same time, the right-hand side of Eq. (3.8) has a finite limit as $\alpha \to 1$, which is equal to integral (3.6) for $\alpha = 1$:

$$I_\upsilon(\varphi_*, 1) = 2\operatorname{Arch}\left(\left(\cos\frac{\varphi_*}{2}\right)^{-1}\right) = 2\left[\ln\left(1+\sin\frac{\varphi_*}{2}\right) - \ln\left(\cos\frac{\varphi_*}{2}\right)\right] \tag{3.9}$$

It follows from this that $I_\upsilon(\varphi_*, 1) \to 0$ as $\varphi_* \to 0$. It can be established that $I_\upsilon \to \infty$ when $\varphi_* \to \pi$. This property of the unboundedness of the oscillation period Θ_υ is characteristic of motions close to the separatrix ($E \lesssim 2$). On the other hand, for small oscillation amplitudes ($\varphi_* \ll 1$, $E \ll 1$), for the integral I_υ and the period Θ_υ, we obtain the approximate expressions

$$I_\upsilon \approx \frac{\pi}{\sqrt{2}}\sqrt{1-\alpha}, \quad \Theta_\upsilon \approx 2\pi\sqrt{1-2\lambda} \tag{3.10}$$

These quantities are independent of φ_* (or E); the quantity Θ_υ in system (3.10) is identical with the period of small oscillations of the cylinder described by linearized equation (3.2) with $\varepsilon = 0$.

When $\alpha \to 0$, the parameter p approaches zero and the complete elliptic integral of the third kind $\Pi((\pi/2), p, \kappa)$ becomes a complete elliptic integral of the first kind $\mathbf{K}(\kappa)$.

The integral $I_r(\beta, \alpha)$, related to the rotation period Θ_r, in expressions (3.7) also requires lengthy calculations. It can be reduced to the form

$$I_r(\beta,\alpha) = \frac{2(1-\alpha)}{\sqrt{(1+\alpha)(1+\beta)}}\Pi\left(\frac{\pi}{2}, p, \kappa\right)$$
$$p = \frac{2\alpha}{1+\alpha}, \quad \kappa = \sqrt{\frac{2(\alpha+\beta)}{(1+\alpha)(1+\beta)}} \tag{3.11}$$

As in expression (3.8) for I_υ, the coefficient of the integral $\Pi((\pi/2), p, \kappa)$ in Eq. (3.11) approaches zero as $\alpha \to 1$ ($\lambda \to (1/2)$), and the integral itself increases without limit. However, when $\alpha \to 1$, the right-hand side of Eq. (3.11) approaches a finite limit

$$I_r(\beta, 1) = \frac{2}{\sqrt{\beta}}\operatorname{Arsh}\left(\sqrt{\frac{2\beta}{1-\beta}}\right), \quad 0<\beta<1$$
$$I_r(\beta,1) \to 2\sqrt{2}, \quad \beta \to 0; \quad I_r(\beta, 1) \to \infty, \quad \beta \to 1 \tag{3.12}$$

In the general case $0 \le \alpha \le 1$, the quantities I_r (3.11), (3.12) and Θ_r (3.7) are unbounded as $\beta \to 1$, which corresponds to slow rotational motions close to the separatrix $E \gtrsim 2$. For the period of rapid rotations ($E \gg 1$, $\beta \ll 1$), the following approximate expression applies

$$\Theta_r(E,\lambda) \approx 2\sqrt{\frac{2(1+2\lambda)}{E}}\mathbf{E}\left(\sqrt{\frac{1+2\lambda}{2}}\right) \ll 1 \tag{3.13}$$

where $\mathbf{E}$ is the complete elliptic integral of the second kind. Note that, when $\alpha=0$, the parameter p vanishes, and the integral $\Pi((\pi/2), p, \kappa)$ becomes a complete elliptic integral of the first kind $\mathbf{K}(\kappa)$ with an inherent property of divergence as $\kappa \to 1$, i.e. as $\beta \to 1$.

To describe the dynamics of system (3.2) when $\varepsilon > 0$, it is convenient to proceed from the coordinate–velocity phase variables (φ, φ') to the energy–phase variables (E, Ψ). The variable E is defined by relation (3.5) and varies as given by the equation

$$E' = -\varepsilon u \varphi'(E,\varphi) \tag{3.14}$$

which is obtained by differentiating expression (3.5) by virtue of Eq. (3.2). The function $\varphi'(E, \varphi)$ is implicity defined by relation (3.5). Solving this relation for φ', we obtain

$$\varphi'(E,\varphi) = \pm\left[\frac{2(E-1+\cos\varphi)}{1-2\lambda\cos\varphi}\right]^{\frac{1}{2}} \tag{3.15}$$

The phase Ψ is defined by the relation

$$\Psi = \frac{2\pi}{\Theta(E)}\int_{\varphi_0}^{\varphi}\frac{d\xi}{\varphi'(E,\xi)} \tag{3.16}$$

where φ_0 is the value of the variable φ when $\theta = 0$. Differentiating expression (3.16) with respect to θ, taking Eq. (3.14) into account, we obtain

$$\Psi' = \frac{2\pi}{\Theta(E)} - \varepsilon\frac{\partial}{\partial E}\left(\frac{2\pi}{\Theta(E)}\int_{\varphi_0}^{\varphi}\frac{d\xi}{\varphi'(E,\xi)}\right)\varphi'(E,\varphi)u \tag{3.17}$$

Expressing the variable φ as a function of E and Ψ with reference to relation (3.16), and substituting $\varphi = \varphi(E, \Psi)$ into the right-hand sides of Eqs. (3.14) and (3.17), we obtain a closed system of differential equations for the variables E and Ψ. The right-hand sides of this system are 2π-periodic functions of Ψ.

In the first approximation of the method of averaging,[4,5] the phase Ψ is found with an error $O(1)$ on the time interval $\theta \sim \varepsilon^{-1}$, since the slow variable (the energy E) is calculated with an error $O(\varepsilon)$ on this interval. Therefore, the term with the coefficient ε in Eq. (3.17) can be omitted without loss of accuracy, which considerably smiplifies the solution of control problems for the mechanical system considered.

4. Near-optimal modes for controlling the energy of oscillations and rotation of an inhomogeneous cylinder

Let us consider optimal control problems for system (3.14), (3.17) in an asymptotically long time interval $(\theta \sim \varepsilon^{-1})$.[1,2]

Problem 1. *Maximization or minimization of the total energy at a prescribed instant of time*. For the system (3.14), (3.17) subject to the initial conditions

$$E(0) = E_0, \quad \Psi(0) = \Psi_0 \tag{4.1}$$

it is required to find the optimal control $u = u_0(E, \Psi)$ which satisfies the constraint $|u| \le 1$ and minimizes (maximizes) the energy E at a specified instant of time $\theta_f = L\varepsilon^{-1}$, i.e.

$$J_1(u) = \pm E(\theta_f) \to \min_u, \quad |u| \le 1, \quad \theta_f = L\varepsilon^{-1} \tag{4.2}$$

Problem 2. *Driving the system to a state with specified energy in the minimum time.* It is required to find the optimal control $u = u_0(E, \Psi)$ which satisfies the constraint $|u| \le 1$ and which brings system (3.14), (3.17), subject to initial conditions (4.1), to a state with energy $E = E_f$ in the minimum time, i.e.

$$J_2(u) = \theta_f \to \min_u, \quad |u| \le 1, \quad E(\theta_f) = E_f \tag{4.3}$$

An approximate solution of these problems can be constructed using Pontryagin's maximum principle,[3] combined with the method of averaging,[4,5] which is used to solve the boundary value problem of the maximum principle.[1,2] To fix our ideas, we will dwell on Problem 2. Approximate (near-optimal) control is represented in a feedback form as follows:

$$u_0 = \operatorname{sign}[(E - E_f)\varphi'], \quad E \ne E_f \tag{4.4}$$

The control (4.4) is locally optimal, and it is this control that, at any instant of time, minimizes the rate of change of the modulus of the difference between the actual value of the energy E and the final value E_f.

Note[1,2] that, for Problem 1, the near-optimal control has the form $u_0 = \mp\operatorname{sign}\varphi'$.

The symbol φ' in relation (4.4) should be understood as the function (3.15), where the quantity φ is expressed in terms of E and Ψ in accordance with relation (3.16). Thus, the control (4.4) is a function of the variables E and Ψ, as required by the formulation of the optimal control problem. On the other hand, this control can be expressed as a function of the initial variables φ and φ' of system (3.2), which can be measured directly. For this it is necessary to substitute expression (3.5) for E into Eq. (4.4).

The change in the variable E with an error $O(\varepsilon)$ on the time interval $\sim\varepsilon^{-1}$ is described by the equation obtained after substituting the control (4.4) into relation (3.14) and averaging the latter over the phase Ψ. The introduction of the slow argument $\tau = \varepsilon\theta$ makes it possible to reduce the averaged equation to the form (the dot denotes differentiation with respect to τ)

$$\begin{gathered} E^{\bullet} = V(E, \lambda)\operatorname{sign}(E - E^f), \quad E(0) = E_0 \\ V(E, \lambda) = \frac{1}{2\pi}\int_0^{2\pi}|\varphi'(E, \Psi)|d\Psi = \frac{1}{\Theta(E, \lambda)}\int_0^{\Theta}|\varphi'|d\theta = \frac{1}{\Theta}\oint \operatorname{sign}\varphi' d\varphi, \quad \Theta = \Theta_{\upsilon, r} \end{gathered} \tag{4.5}$$

The dependence of the oscillation period Θ_υ or rotation period Θ_r on E and λ is defined by expressions (3.6), (3.10) or (3.7), (3.13).

The integrals in the expression for $V(E, \lambda)$ are evaluated analytically, and for the function V yield the expressions

$$\begin{aligned} V_\upsilon(E, \lambda) &= \frac{4}{\Theta_\upsilon(E, \lambda)}\varphi_*(E) = \frac{4\arccos(1 - E)}{\Theta_\upsilon(E, \lambda)}, \quad 0 < E < 2 \\ V_r(E, \lambda) &= \frac{2\pi}{\Theta_r(E, \lambda)}, \quad E > 2 \end{aligned} \tag{4.6}$$

The amplitude of the oscillations φ_* is given by the second formula of system (3.6); for $E \ll 1$, we have $\varphi_* \approx \sqrt{2E}$. Equation (4.5) allows of the separation of variables, and its solution is expressed by the quadrature

$$\tau = \int_{E_0}^{E} \frac{dE}{\operatorname{sign}(E - E_f)V(E, \lambda)} \tag{4.7}$$

The minimum slow time of variation of the energy from E_0 to E_f is calculated from the formula

$$\tau_f = \left|\int_{E_0}^{E_f} \frac{dE}{V(E, \lambda)}\right|, \quad 0 < \lambda < \frac{1}{2}, \quad 0 \le E_0, \quad E_f < \infty \tag{4.8}$$

An analysis of the properties of function V indicates the complete controllability of the system in the oscillation and rotation modes. When $E \ll 1$ and $E \gg 1$, we have the asymptotic form $V\ \sqrt{E}$, i.e. $E \sim \tau^2$. The control (4.4) is also applicable in the case of the evolution of the motion of the cylinder from the oscillation mode to the rotation mode. Transition through the separatrix leads to an increase in the degree of error of the asymptotic solution: instead of the estimate of $O(\varepsilon)$, we have the estimate $O(\varepsilon \ln \varepsilon^{-1})$, which is satisfactory from the viewpoint of applications for sufficiently small $\varepsilon > 0$.

5. Mathematical modelling of the controlled oscillations and rotations of an inhomogeneous cylinder with a near-optimal control law

The near-optimal control (4.4), constructed in the feedback form, enables one to control system (3.2), continuously measuring its phase variables φ and φ' and calculating the energy E by means of formula (3.5). In this section, the results of numerical simulation of the behaviour of the system under the given control law are presented and analysed. The simulation involve the integrating the equation

$$\begin{aligned} &(1-2\lambda\cos\varphi)\varphi'' + (1+\lambda\varphi'^2)\sin\varphi = -\varepsilon u_0(\varphi, \varphi') \\ &u_0(\varphi, \varphi') = \operatorname{sign}[(E(\varphi, \varphi') - E_f)\varphi'] \\ &E = \frac{1}{2}(1-2\lambda\cos\varphi)\varphi'^2 + 1 - \cos\varphi \end{aligned} \tag{5.1}$$

with prescribed parameters λ, ε and E_f, and the initial conditions

$$\varphi(0) = \varphi_0, \quad \varphi'(0) = \varphi_0' \tag{5.2}$$

Equation (5.1) was obtained by substituting the near-optimal control (4.4) into Eq. (3.2). The right-hand side of this equation is a discontinuous function of the phase variables, and here the number of discontinuities along the trajectory of motion can be asymptotically large ($\sim\varepsilon^{-1}$). This fact considerably complicates the numerical simulation of Eq. (5.1) using standard software. Therefore, it is reasonable to smooth the control function (4.4) on a set of discontinuities $(E - E_f)\varphi' = 0$, thereby smoothing the right-hand side of Eq. (5.1).

When simulating the motions of the system, the results of which are presented below, the control (4.4) was approximated by the function

$$u_*(\varphi, \varphi') = \frac{E - E_f}{|E - E_f| + d_E}\,\frac{\varphi' + c}{|\varphi'| + d_\varphi} \tag{5.3}$$

where d_E, d_φ and c are parameters of the approximation, d_E and d_φ being positive. As all the approximation parameters approach zero, the function u_* [Eq. (5.3)] approaches the function u_0 [Eq. (4.4) or Eq. (5.1)]. It is assumed that $|c| \le d_\varphi$. This condition ensures that the constraint $|u_*| \le 1$, imposed on the control variable, is satisfied. The parameters d_E and d_φ are introduced to smooth the bang-bang function. The additional parameter c is introduced to ensure non-zero control when $\varphi' = 0$, for example, at the start of motion from an equilibrium state. By appropriate choice of the sign of c it is possible to control the direction of motion on the initial segment, and also on passing through the separatrix (at near-zero velocity) in the case when $E_0 < 2$ but $E_f > 2$. If $c = 0$, the direction of motion of the cylinder on passing through the separatrix proves to be difficult to predict, especially for extremely small values of ε. The specified energy $E_f > 2$ is reached irrespective of the direction of rotation. Furthermore, in the case of the control (5.2) with $c = 0$, considerable influence of rounding-off errors is observed when integrating the equation of motion (3.2) at the stage of the transition from oscillations to rotations. This also leads to indeterminacy in the direction of rotation at the final stage of the motion. For the deceleration process (when $E_f < E_0$), it is possible to assume that $c = 0$ irrespective of the oscillatyion or rotation mode.

Thus, when simulating, instead of Eq. (5.1), we integrated the equation

$$(1-2\lambda\cos\varphi)\varphi'' + (1+\lambda\varphi'^2)\sin\varphi = -\varepsilon u_*(\varphi, \varphi') \tag{5.4}$$

The input data for the simulation included the dimensionless parameters of the system λ, ε and ζ [see system (3.3)], the coefficient of friction of the cylinder against the rough surface k, the initial energy E_0 and the final energy E_f and also the approximating control parameters d_E, d_φ and c. The initial conditions were specified as follows:

$$\varphi_0 = 0, \quad \varphi_0' = \sqrt{\frac{2E_0}{1-2\lambda}} \tag{5.5}$$

The first condition means that, in the initial state, the centre of mass of the cylinder occupies the lower position. The initial velocity φ_0' was calculated by means of formula (3.15) for $E=E_0$ and $\varphi=\varphi_0=0$; to be specific, the right-hand sign was taken with a plus sign.

The output data contained the functions $\varphi(\theta)$, $\varphi'(\theta)$, $E(\theta)$, $u_*(\varphi(\theta), \varphi'(\theta))$, $N^*(\theta)$ and $\mathcal{F}(\theta) = |F_*(\theta)| - kN^*(\theta)$. They characterize the dependence on the dimensionless time θ of the dimensionless values of the angle of rotation of the cylinder, its angular velocity, energy E, control (5.5), the normal reaction of the plane on the cylinder and the difference between the modulus of the friction force of the cylinder against the plane and the normal reaction multiplied by the coefficient of friction. The quantities N^* and F^* are defined by formulae (3.3). In the calculations it was assumed that

$$\zeta = 0.5, \quad k = 0.5, \quad d_E = 0.01, \quad d_\varphi = 0.01, \quad c = 0.001$$

The values of ε, λ, E_0 and E_f underwent changes. Graphs of the functions mentioned are presented in Figs. 2–5 for four sets of initial data.

The curves in Figs. 2–4 correspond to the values $\lambda=0.05$, $\varepsilon=0.1$ and E_0 and to different values of E_f: $E_f=1.5$ (Fig. 2), $E_f=5$ (Fig. 3), and $E_f=2$–5.576×10^{-6} (Fig. 4). The version with $E_f=1.5$ corresponds to swinging up the cylinder from stable equilibrium to the angular amplitude $\varphi_0=120°$, calculated by means of the formula $\varphi_0=\arccos(1-E)$ for $E=1.5$. The required time $\theta_f\approx 27.6$ corresponds to formula (4.8). When $E_f=5$, the cylinder is driven from a state of stable equilibrium into rotation. The next-minimum time of operation $\theta_f\approx 38$ agrees with expression (4.8).

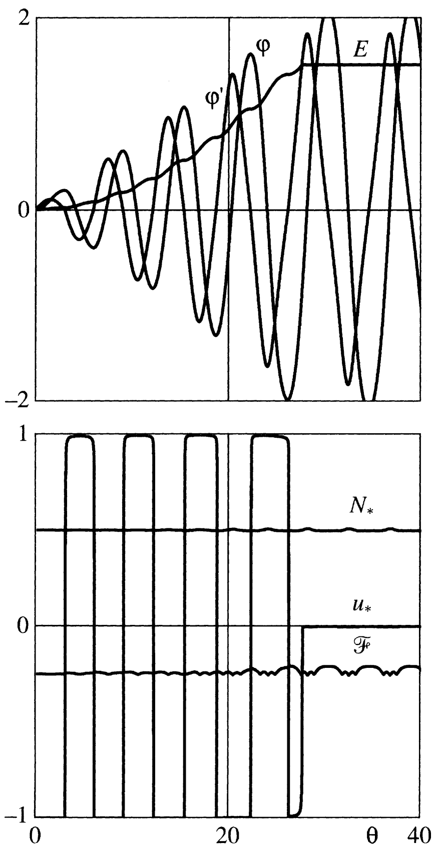

Fig. 2.

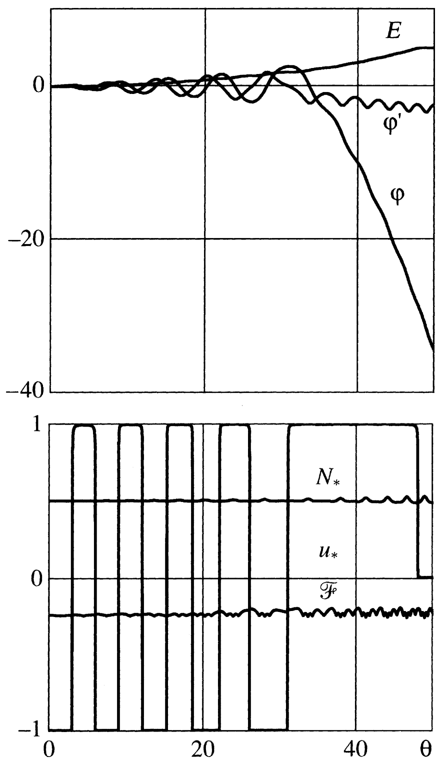
E
0
φ'
φ
−20
−40
1
N_*
u_*
0
$\mathcal{F}$
−1
0
20
40
θ

Fig. 3.

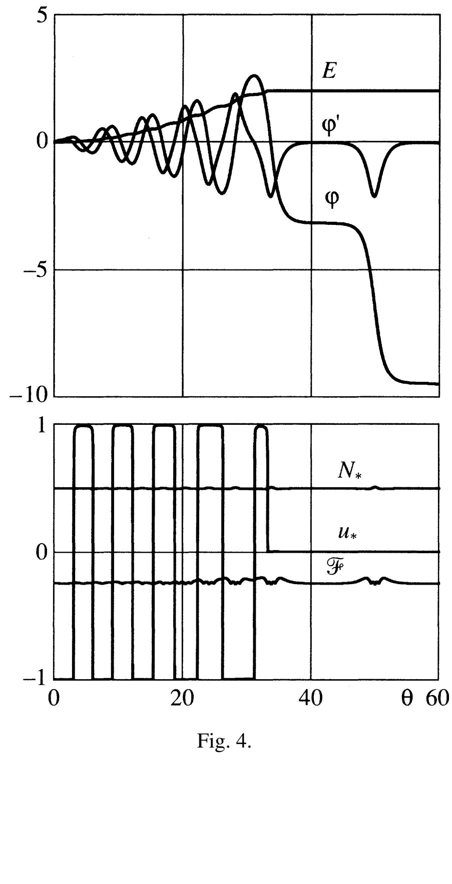
5
E
φ'
0
φ
−5
−10
1
N_*
u_*
0
$\mathcal{F}$
−1
0
20
40
θ 60

Fig. 4.

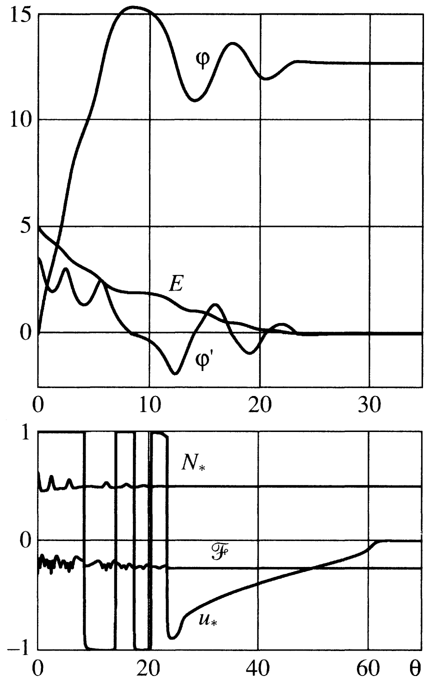

Fig. 5.

When $E_f = 2–5.576 \times 10^{-6}$, the next-minimum time of operation $\theta_f \approx 33$. This case corresponds to the transfer of the cylinder from stable equilibrium, characterized by the value $\varphi = 0$, to unstable equilibrium with $\varphi = \pi(\text{mod}2\pi)$ or $\varphi = -\pi(\text{mod}2\pi)$. On the phase plane $\varphi\varphi'$ this equilibrium is represented by the point $(\pi, 0)$ or $(-\pi, 0)$ on the separatrix separating the regions of oscillational and rotational motions. For the phase states lying on the separatrix, $E_f = 2$ (see Section 3). However, with this value of the final energy, on account of computing errors and instability of the equilibrium into which it is required to bring the system, the simulation results do not indicate any marked "stay" of the cylinder in the unstable equilibrium position. To demonstrate the effect of the stay, an E_f value close to $E_f = 2$ was selected experimentally. These experiments showed the very high sensitivity of the simulation of the effect of stay in the unstable equilibrium position to a variation in the value of the energy and to the choice of the step of numerical integration of the equation of motion.

Fig. 5 presents curves for the initial data

$$\lambda = 0.1, \quad \varepsilon = 0.2, \quad E_0 = 5, \quad E_f = 0$$

In this case, deceleration of the initial rotation of the cylinder to a complete stop at a stable equilibrium position occurs. The approximate time of deceleration $\theta_f \approx 24$.

For all versions of the simulation, the curve $\mathcal{F}(\theta)$ lies below the abscissa axis, which indicates that conditions (3.3) are satisfied, thereby guaranteeing motion of the cylinder over the plane without loss of contact or slippage. Accordingly, the curve $N^*(\theta)$ lies above the abscissa axis.

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